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A CONTINUUM THEORY FOR FIBRE-REINFORCED COMPOSITES

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Abstract—The effective stiffness theory for fibre reinforced materials with a hexagonal layout of fibres is presented. The theory is illustrated by the dispersion curves of plane steadystate time-harmonic waves. The limiting phase velocities at vanishing wave numbers serve in the determination of the elastic moduli of the equivalent homogeneous transversely isotropic medium.

1. INTRODUCTION

In Refs. [1, 2] Sun, *et al.* developed a continuum theory for a laminated medium, which they named "the effective stiffness theory". There the actual composite was transformed into a homogeneous higher-order continuum with microstructure.

Achenbach and Herrmann[3] presented a simple continuum theory for a unidirectional fibre-reinforced composite. In their model the fibres are endowed with stiffnesses against flexure, torsion and extension. The fibres are embedded in a fictitious transversely isotropic matrix. The elastic constants of this matrix are determined from the condition that the phase velocities of plane harmonic waves—if the wave lengths tend to infinity—should equal those velocities in the equivalent homogeneous material. It is therefore necessary to know the effective moduli. In this model the displacements are not continuous at the interfaces and only the transverse wave propagating in the direction of the fibres is dispersive.

A mixture theory modelling wave propagation in laminated and unidirectional fibrous composites is presented in Ref. [9]. It considers only the case of gross-displacements parallel to laminates and fibres.

In Refs. [10, 11] the effective stiffness theory is applied for the case of fibres arranged in rectangular arrays and the propagation of plane waves is studied. In Ref. [10] only continuity in the mean is required at the interfaces of the neighbouring cells. In Ref. [11] the displacement in the matrix inside the cell is smooth only by parts.

In this paper we shall evolve the effective stiffness theory for a unidirectional fibre-reinforced composite with fibres arranged hexagonally. The geometry of the composite is described in Section 2. The interaction between the fibre and the matrix and between the neighbouring cells is taken into account by simulating point by point continuity of the displacements at the interfaces. In Section 3, Hamilton's principle is employed to obtain the stress equations of motion, the constitutive equations and the displacement equations of motion. Section 4 studies the propagation of plane harmonic waves in the composite. The waves are dispersive and the dispersion curves are compared with those reported in Refs. [3, 7]. A good agreement is found to exist for long waves. It is established that the structure of plane harmonic waves in this model is the same as that for a homogeneous transversely isotropic continuum, the latter showing no dispersion. For statical problems we propose to replace the effective stiffness model by a homogeneous transversely isotropic medium the elastic moduli of which are determined in

Section 5 by comparing the phase velocities in the homogeneous model with the phase velocities at vanishing wave numbers in the effective stiffness model.

2. KINEMATICS

Let us consider a material consisting of two components: matrix and fibres. Both the matrix and the fibres are linear elastic homogeneous and isotropic materials. The Lamé constants and the mass density of the fibres and of the matrix are denoted by λ_1, μ_1, ρ_1 and λ_2, μ_2, ρ_2, respectively. The infinitely long fibres are of circular cross-section with radius r_1 and are arranged in a hexagonal array throughout the matrix material (see Fig. 1a). The fibres are parallel to the

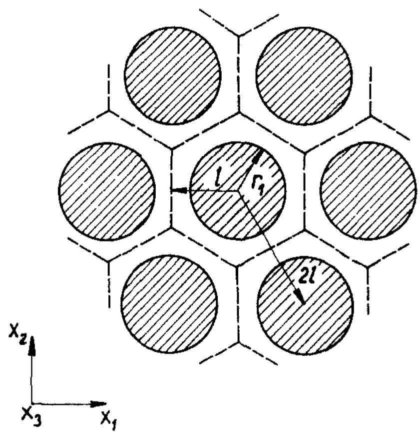

Fig. 1a. Uni-directionally fibre-reinforced composite with hexagonal array.

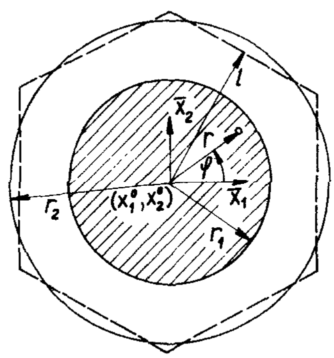

Fig. 1b. Representative element.

x_3-axis and the distance between them is $2l$. A perfect bond is assumed to exist between the two materials. The regular hexagonal prisms in Fig. 1a are replaced by circular cylinders of the same volume (Fig. 1b). The radius of the cylinders then is

$$r_2 = l\sqrt{\frac{2\sqrt{3}}{\pi}}.$$

This composite cylinder will be referred to as the representative element.

Let x_1, x_2, x_3 be the global Cartesian coordinates. Let us introduce $\bar{x}_1$, $\bar{x}_2$, $\bar{x}_3$ as the local Cartesian coordinates in the representative element with

$$x_1 = x_1^0 + \bar{x}_1, \quad x_2 = x_2^0 + \bar{x}_2, \quad x_3 = \bar{x}_3,$$

where x_1^0, x_2^0 are the global coordinates of the axis of the representative element. Let r, φ denote the local polar coordinates (Fig. 1b), i.e.

$$\bar{x}_1 = r \cos \varphi, \quad \bar{x}_2 = r \sin \varphi.$$

We assume a linear dependence of the vector of displacement in the fibre, ${}^1u_i$, on $\bar{x}_1$, $\bar{x}_2$ and a linear dependence of the vector of displacement in the matrix, ${}^2u_i$, on r. We write ${}^1u_i$, ${}^2u_i$ in the form

$${}^1u_i(x_j, t) = {}^1u_i^0(x_1^0, x_2^0, x_3, t) + r\,{}^1U_i^0(x_1^0, x_2^0, x_3, \varphi, t), \tag{2.1}$$

$${}^2u_i(x_j, t) = {}^2u_i^0(x_1^0, x_2^0, x_3, r_2, \varphi, t) + (r - r_2)\,{}^2U_i^0(x_1^0, x_2^0, x_3, \varphi, t). \tag{2.2}$$

Here ${}^1u_i^0(x_1^0, x_2^0, x_3, t)$ is the displacement in the axis of the fibre, ${}^2u_i^0(x_1^0, x_2^0, x_3, r_2, \varphi, t)$ is the displacement on the surface of the representative element at the point with the local coordinates r_2, φ, x_3. ${}^1u_i^0, {}^2u_i^0, {}^1U_i^0, {}^2U_i^0$ are discrete functions with respect to x_1^0, x_2^0. The derivation of a continuum theory calls for a smoothing operation. We replace the discrete variables x_1^0, x_2^0 by the continuous variables x_1, x_2 with

$${}^1u_i^0(x_j, t) = {}^2u_i^0(x_j, 0, \varphi, t) \equiv u_i(x_j, t)$$

and call u_i "the gross-displacement". As ${}^1u_i$, ${}^2u_i$ depend linearly on $\bar{x}_1$, $\bar{x}_2$ we get for ${}^1U_i^0$ that

$${}^1U_i^0(x_j, \varphi, t) = \psi_{1i}(x_j, t) \cos \varphi + \psi_{2i}(x_j, t) \sin \varphi, \tag{2.3}$$

where

$$\psi_{1i}(x_j, t) \equiv {}^1U_i^0(x_j, 0, t), \quad \psi_{2i}(x_j, t) \equiv {}^1U_i^0\left(x_j, \frac{\pi}{2}, t\right).$$

Neglecting the higher powers of r_2, we can write

$$u_i(x_1 + r_2 \cos \varphi, x_2 + r_2 \sin \varphi, x_3, t) = u_i(x_j, t) + r_2[u_{i,1}(x_j, t) \cos \varphi + u_{i,2}(x_j, t) \sin \varphi]. \tag{2.4}$$

Here a comma followed by an index denotes partial differentiation with respect to the corresponding Cartesian coordinate. The condition of continuity of the displacement at the interfaces of the fibre gives from (2.1), (2.2) and (2.4) the following dependence of ${}^2U_i^0$ on φ:

$$(r_2 - r_1)\,{}^2U_i^0(\varphi) = r_2(u_{i,1} \cos \varphi + u_{i,2} \sin \varphi) - r_1\,{}^1U_i^0(\varphi). \tag{2.5}$$

Here and in what follows we shall omit the dependence of the functions on x_j and time t. Using (2.3), (2.4) and (2.5) we get (2.1), (2.2) in the form

$${}^1u_i = u_i + \bar{x}_1\psi_{1i} + \bar{x}_2\psi_{2i}, \tag{2.6}$$

$$ {}^2u_i = u_2 + \cos\varphi\left[r_2u_{i,1} + \frac{r-r_2}{r_2-r_1}(r_2u_{i,1} - r_1\psi_{1i})\right] + \sin\varphi\left[r_2u_{i,2} + \frac{r-r_2}{r_2-r_1}(r_2u_{i,2} - r_1\psi_{2i})\right]. \quad (2.7) $$

The state of deformation in the medium is now described by the gross-displacement u_i and by the quantities ψ_{1i}, ψ_{2i}, the latter describing local deformations in the representative element.

3. THE BASIC EQUATIONS

Let V be a fixed regular region of the medium. Hamilton's principle for independent variations of u_i, ψ_{1i}, ψ_{2i} may be written as

$$ \delta\int_{t_1}^{t_2}\int_V (K - W)\,\mathrm{d}V\,\mathrm{d}t = 0. \quad (3.1) $$

In the above equation K is the kinetic energy and W is the strain energy per unit volume of the medium. In this paper we are interested in the propagation of harmonic waves through an infinitely extended medium rather than in the formulation of the boundary—initial value problems. That is why we start with Hamilton's principle in the restricted form (3.1) setting the variations of all kinematic quantities equal to zero on the boundary of V for $t_1 \le t \le t_2$ and in V at times t_1, t_2. To get the boundary and initial conditions we could proceed in the same way as in Refs. [5] or [1].

The strain energy density is defined by

$$ W = \frac{1}{\pi r_2^2}\left(\int\int_{{}^1F} {}^1w\,\mathrm{d}\bar{x}_1\,\mathrm{d}\bar{x}_2 + \int\int_{{}^2F} {}^2w\,\mathrm{d}\bar{x}_1\,\mathrm{d}\bar{x}_2\right), $$

$$ {}^1w = \frac{1}{2}\lambda_1\,{}^1\epsilon_{ii}\,{}^1\epsilon_{kk} + \mu_1\,{}^1\epsilon_{ij}\,{}^1\epsilon_{ij}, \quad {}^2w = \frac{1}{2}\lambda_2\,{}^2\epsilon_{ii}\,{}^2\epsilon_{kk} + \mu_2\,{}^2\epsilon_{ij}\,{}^2\epsilon_{ij}, \quad (3.2) $$

$$ {}^1\epsilon_{ij} = {}^1u_{(i,j)}, \quad {}^2\epsilon_{ij} = {}^2u_{(i,j)}. $$

Here differentiation is taken with respect to the local coordinates $\bar{x}_1$, $\bar{x}_2$, x_3. Summation of pairs of identical indices over 1, 2, 3 is implied. The first integral in (3.2) is taken over the part of the cross-section of the representative element belonging to the fibre, the second over that belonging to the matrix. We can now calculate W using (2.6), (2.7). We get the following compact expression for W:

$$ W = \frac{1}{2}A_{ijkl}\epsilon_{ij}\epsilon_{kl} + \frac{1}{2}B_{ijkl}\gamma_{kl}\epsilon_{ij} + \frac{1}{2}C_{ijkl}\gamma_{ij}\gamma_{kl} + \frac{1}{2}D_{ijklmn}\mathcal{H}_{ijk}\mathcal{H}_{lmn} + \frac{1}{2}E_{ijklmn}\mathcal{H}_{ijk}\vartheta_{lmn} + \frac{1}{2}F_{ijklmn}\vartheta_{ijk}\vartheta_{lmn} \quad (3.3) $$

with

$$ A_{1111} = A_{2222} = A_{3333} = \eta^2(\lambda_1 + 2\mu_1) + (1-\eta^2)(\lambda_2 + 2\mu_2), $$

$$ A_{1212} = A_{2112} = A_{1221} = A_{2121} = A_{1313} = A_{3113} = A_{1331} = A_{3131} = A_{2323} = A_{3223} = A_{2332} = A_{3232} = \eta^2\mu_1 + (1-\eta^2)\mu_2, $$

$$ A_{1122} = A_{2211} = A_{1133} = A_{3311} = A_{2233} = A_{3322} = \eta^2\lambda_1 + (1-\eta^2)\lambda_2, $$

$$ B_{1111} = B_{2222} = 2\eta^2[(\lambda_2 - \lambda_1) + 2(\mu_2 - \mu_1)], $$

$$B_{1122} = B_{2211} = B_{3311} = B_{3322} = 2\eta^2(\lambda_2 - \lambda_1),$$
$$B_{1212} = B_{2112} = B_{1221} = B_{2121} = B_{1313} = B_{3113} = B_{2323} = B_{3223} = 2\eta^2(\mu_2 - \mu_1),$$

$$C_{1111} = C_{2222} = \eta^2(\lambda_1 + 2\mu_1) + (3V - \eta^2)\lambda_2 + (7V - 2\eta^2)\mu_2,$$
$$C_{1212} = C_{2121} = \eta^2\mu_1 + V\lambda_2 + (5V - \eta^2)\mu_2,$$
$$C_{1221} = C_{2112} = \eta^2\mu_1 + V\lambda_2 + (V - \eta^2)\mu_2,$$
$$C_{1122} = C_{2211} = \eta^2\lambda_1 + (V - \eta^2)\lambda_2 + V\mu_2,$$
$$C_{1313} = C_{2323} = \eta^2\mu_1 + (4V - \eta^2)\mu_2,$$

$$D_{313313} = D_{323323} = \frac{r_1^2}{4}\left[\eta^2(\lambda_1 + 2\mu_1) + \frac{1-\eta^4}{\eta^2}(\lambda_2 + 2\mu_2)\right],$$
$$D_{311311} = D_{312312} = D_{321321} = D_{322322} = \frac{r_1^2}{4}\left(\eta^2\mu_1 + \frac{1-\eta^4}{\eta^2}\mu_2\right),$$

$$E_{313313} = E_{323323} = r_1^2\frac{3-\eta-\eta^2-\eta^3}{6\eta^2}(\lambda_2 + 2\mu_2),$$
$$E_{311311} = E_{312312} = E_{321321} = E_{322322} = r_1^2\frac{3-\eta-\eta^2-\eta^3}{6\eta^2}\mu_2,$$

$$F_{313313} = F_{323323} = r_1^2\frac{3-2\eta-\eta^2}{12\eta^2}(\lambda_2 + 2\mu_2),$$
$$F_{311311} = F_{312312} = F_{321321} = F_{322322} = r_1^2\frac{3-2\eta-\eta^2}{12\eta^2}\mu_2. \tag{3.4}$$

The other components of tensors A_{ijkl}, B_{ijkl}, C_{ijkl}, D_{ijklmn}, E_{ijklmn}, F_{ijklmn} are zero. In (3.3) ϵ_{ij}, γ_{ij}, $\mathscr{K}_{ijk}$, ϑ_{ijk} are defined as follows:

$$\epsilon_{ij} = u_{(i,j)} \quad \text{for} \quad i,j = 1,2,3$$
$$\gamma_{\alpha j} = u_{j,\alpha} - \psi_{\alpha j}, \quad \vartheta_{3\alpha j} = \gamma_{\alpha j,3}, \quad \mathscr{K}_{3\alpha j} = \psi_{\alpha j,3} \quad \text{for} \quad j = 1,2,3; \quad \alpha = 1,2.$$

The other components of ϵ_{ij}, γ_{ij}, ϑ_{ijk}, $\mathscr{K}_{ijk}$ are zero. In (3.4) it is

$$\eta = \frac{r_1}{r_2}, \qquad V = -\frac{1}{4}\frac{\eta^2}{(1-\eta)^2}\lg\eta. \tag{3.5}$$

It is seen that the form of W in (3.3) is the same as that in eq. (7) of [2].

The kinetic energy density K is defined by

$$K = \frac{1}{\pi r_2^2}\sum_{i=1}^{3}\left(\frac{1}{2}\int\int_{{}^1F} \rho_1{}^1\dot{u}_i^2\,d\bar{x}_1\,d\bar{x}_2 + \frac{1}{2}\int\int_{{}^2F} \rho_2{}^2\dot{u}_i^2\,d\bar{x}_1\,d\bar{x}_2\right)$$

and with the aid of (2.6), (2.7) we get

$$K = \frac{1}{2}\sum_{i=1}^{3}\left\{\bar{\rho}\dot{u}_i^2 + J(\dot{\psi}_{1i}^2 + \dot{\psi}_{2i}^2) + \rho_2\left[\frac{X}{2}(\dot{u}_{i,1}^2 + \dot{u}_{i,2}^2) + Z(\dot{u}_{i,1}\dot{\psi}_{1i} + \dot{u}_{i,2}\dot{\psi}_{2i})\right]\right\}, \tag{3.6}$$

where

$$\bar{\rho} = \eta^2\rho_1 + (1-\eta^2)\rho_2, \quad X = r_1^2\frac{3-2\eta-\eta^2}{6\eta^2}, \quad J = \frac{r_1^2}{4}\left[\rho_1\eta^2 + \frac{\rho_2}{3}(1+2\eta-3\eta^2)\right], \quad Z = r_1^2\frac{1-\eta^2}{6\eta}.$$

From now on we proceed analogously as [2] for a laminated material. We define the stress tensors

$$\tau_{ij} = \frac{\partial W}{\partial \epsilon_{ij}}, \quad \sigma_{ij} = \frac{\partial W}{\partial \gamma_{ij}}, \quad \mu_{ijk} = \frac{\partial W}{\partial \mathscr{K}_{ijk}}, \quad \chi_{ijk} = \frac{\partial W}{\partial \vartheta_{ijk}} \tag{3.7}$$

and calculate the variations δK, δW. With their substitution (3.1) yields the following stress equations of motion

$$\begin{aligned} &\tau_{ij,i} + \sigma_{ij,i} - \chi_{kij,ik} - P_j = 0, \quad j = 1,2,3 \\ &\mu_{kij,k} - \chi_{kij,k} + \sigma_{ij} - Q_{ij} = 0, \quad i = 1,2; \quad j = 1,2,3 \end{aligned} \tag{3.8}$$

where

$$P_i = \bar{\rho}\ddot{u}_i - \frac{1}{2}\rho_2[X(\ddot{u}_{i,11} + \ddot{u}_{i,22}) + Z(\ddot{\psi}_{1i,1} + \ddot{\psi}_{2i,2})]$$

$$Q_{\alpha i} = J\ddot{\psi}_{\alpha i} + \frac{1}{2}\rho_2 Z\ddot{u}_{i,\alpha}, \qquad i = 1,2,3; \quad \alpha = 1,2.$$

Substituting (3.7) and (3.3) into (3.8) gives the displacement equations of motion

$$\begin{aligned} &a_1u_{1,11} + a_2u_{1,22} + a_3u_{1,33} + a_4u_{2,12} + a_5u_{3,13} + a_6(u_{1,1313} + u_{1,2323}) + a_7\psi_{11,1} + a_8(\psi_{12,2} + \psi_{22,1}) + a_9\psi_{21,2} \\ &\quad + a_{10}\psi_{13,3} + a_{11}(\psi_{11,313} + \psi_{21,323}) + a_{12}\ddot{u}_1 + a_{13}(\ddot{u}_{1,11} + \ddot{u}_{1,22}) + a_{14}(\ddot{\psi}_{11,1} + \ddot{\psi}_{21,2}) = 0, \end{aligned} \tag{3.9}$$

$$\begin{aligned} &a_{15}u_{3,33} + a_{16}(u_{3,11} + u_{3,22}) + a_5(u_{1,13} + u_{2,23}) + a_{17}(u_{3,1313} + u_{3,2323}) + a_{18}(\psi_{11,3} + \psi_{22,3}) \\ &\quad + a_{19}(\psi_{13,1} + \psi_{23,2}) + a_{20}(\psi_{13,313} + \psi_{23,323}) + a_{12}\ddot{u}_3 + a_{13}(\ddot{u}_{3,11} + \ddot{u}_{3,22}) + a_{14}(\ddot{\psi}_{13,1} + \ddot{\psi}_{23,2}) = 0, \end{aligned} \tag{3.10}$$

$$a_{21}\psi_{11,33} + a_{22}\psi_{11} + a_{23}\psi_{22} - a_7u_{1,1} - a_8u_{2,2} - a_{18}u_{3,3} - a_{11}u_{1,313} + a_{24}\ddot{\psi}_{11} - a_{14}\ddot{u}_{1,1} = 0, \tag{3.11}$$

$$a_{21}\psi_{12,33} + a_{25}\psi_{12} + a_{26}\psi_{21} - a_8u_{1,2} - a_9u_{2,1} - a_{11}u_{2,313} + a_{24}\ddot{\psi}_{12} - a_{14}\ddot{u}_{2,1} = 0, \tag{3.12}$$

$$a_{27}\psi_{13,33} + a_{28}\psi_{13} - a_{10}u_{1,3} - a_{19}u_{3,1} - a_{20}u_{3,313} + a_{24}\ddot{\psi}_{13} - a_{14}\ddot{u}_{3,1} = 0, \tag{3.13}$$

where

$$\begin{aligned} &a_1 = (3V+1)\lambda_2 + (7V+2)\mu_2, && a_2 = V\lambda_2 + (5V+1)\mu_2, \\ &a_3 = \eta^2\mu_1 + (1-\eta^2)\mu_2, && a_4 = (2V+1)(\lambda_2+\mu_2), \\ &a_5 = \lambda_2 + \mu_2, && a_6 = -\frac{X}{2}\mu_2, \\ &a_7 = -V(3\lambda_2 + 7\mu_2), && a_8 = -V(\lambda_2+\mu_2), \\ &a_9 = -V(\lambda_2 + 5\mu_2), && a_{10} = \eta^2(\mu_1 - \mu_2), \\ &a_{11} = -\frac{Z}{2}\mu_2, && a_{12} = -\bar{\rho}, \end{aligned}$$

$$a_{13}=\frac{X}{2}\rho_2, \qquad a_{14}=\frac{Z}{2}\rho_2,$$
$$a_{15}=\eta^2(\lambda_1+2\mu_1)+(1-\eta^2)(\lambda_2+2\mu_2),$$
$$a_{16}=(4V+1)\mu_2, \qquad a_{17}=-\frac{X}{2}(\lambda_2+2\mu_2),$$
$$a_{18}=\eta^2(\lambda_1-\lambda_2), \qquad a_{19}=-4V\mu_2,$$
$$a_{20}=-\frac{Z}{2}(\lambda_2+2\mu_2), \qquad a_{21}=\frac{1}{2}\left(\frac{r_1^2}{2}\eta^2\mu_1+Y\mu_2\right),$$
$$a_{22}=-[\eta^2(\lambda_1+2\mu_1)+(3V-\eta^2)\lambda_2+(7V-2\eta^2)\mu_2],$$
$$a_{23}=-[\eta^2\lambda_1+(V-\eta^2)\lambda_2+V\mu_2], \quad a_{24}=-J,$$
$$a_{25}=-[\eta^2\mu_1+V\lambda_2+(5V-\eta^2)\mu_2],$$
$$a_{26}=-[\eta^2\mu_1+V\lambda_2+(V-\eta^2)\mu_2],$$
$$a_{27}=\frac{1}{2}\left[\frac{r_1^2}{2}\eta^2(\lambda_1+2\mu_1)+Y(\lambda_2+2\mu_2)\right],$$
$$a_{28}=-[\eta^2\mu_1+(4V-\eta^2)\mu_2],$$

where

$$Y=\frac{r_1^2}{6}(1+2\eta-3\eta^2). \tag{3.14}$$

Because the problem is symmetric about the x_3-axis, the remaining four equations of motion can be obtained from (3.9), (3.11), (3.12) and (3.13) on replacing in u_i, $\psi_{\alpha j}$ and their derivatives index 1 by index 2 and the other way around.

4. PROPAGATION OF PLANE HARMONIC WAVES IN A FIBRE REINFORCED COMPOSITE

The displacement equations of motion will be used to study the propagation of plane time-harmonic waves. Because of symmetry about the x_3-axis, we may restrict ourselves—without any loss of generality—to a wave of the form

$$u_i=U_i e^{ik(n_2x_2+n_3x_3-ct)}, \quad \psi_{\alpha j}=\Psi_{\alpha j}e^{ik(n_2x_2+n_3x_3-ct)}.$$

In the above equation U_i, $\Psi_{\alpha i}$ ($i=1,2,3$; $\alpha=1,2$) are constant amplitudes, k is the wave number, c is the phase velocity, and n_i are the components of the unit vector defining the direction of propagation. The nine displacement equations of motion split into two systems for U_i, $\Psi_{\alpha i}$:

System I for U_1, Ψ_{12}, Ψ_{21}, Ψ_{13}:

$$U_1k^2[-n_2^2a_2-n_3^2a_3+k^2n_2^2n_3^2a_6-c^2a_{12}+c^2k^2n_2^2a_{13}]+\Psi_{12}ikn_2a_8$$
$$+\Psi_{21}ik[n_2a_9-k^2n_2n_3^2a_{11}-c^2k^2n_2a_{14}]+\Psi_{13}ikn_3a_{10}=0,$$
$$-U_1ikn_2a_8+\Psi_{12}[-k^2n_3^2a_{21}-c^2k^2a_{24}+a_{25}]+\Psi_{21}a_{26}=0,$$
$$-U_1ikn_3a_{10}+\Psi_{13}[-k^2n_3^2a_{27}-c^2k^2a_{24}+a_{28}]=0,$$
$$U_1ik[-n_2a_9+k^2n_2n_3^2a_{11}+c^2k^2a_{14}]+\Psi_{12}a_{26}+\Psi_{21}[-k^2n_3^2a_{21}-c^2k^2a_{24}+a_{25}]=0.$$

System II for U_2, U_3, Ψ_{11}, Ψ_{22}, Ψ_{23}:

$$U_2k^2[-n_2^2a_1-n_3^2a_3+k^2n_2^2n_3^2a_6-c^2a_{12}+c^2k^2n_2^2a_{13}]-U_3k^2n_2n_3a_5+\Psi_{11}ikn_2a_8$$
$$+\Psi_{22}ik[n_2a_7-k^2n_2n_3^2a_{11}-c^2k^2n_2a_{14}]+\Psi_{23}ikn_3a_{10}=0,$$

$$-U_2k^2n_2n_3a_5 + U_3k^2[-n_3^2a_{15} - n_2^2a_{16} + k^2n_2^2n_3^2a_{17} - c^2a_{12} + c^2k^2n_2^2a_{13}] + \Psi_{11}ikn_3a_{18} + \Psi_{22}ikn_3a_{18} + \Psi_{23}ik[n_2a_{19} - k^2n_2n_3^2a_{20} - c^2k^2n_2a_{14}] = 0,$$

$$-U_2ikn_2a_8 - U_3ikn_3a_{18} + \Psi_{11}[a_{22} - k^2n_3^2a_{21} - c^2k^2a_{24}] + \Psi_{22}a_{23} = 0,$$

$$U_2ik[-n_2a_7 + k^2n_2n_3^2a_{11} + c^2k^2n_2a_{14}] - U_3ikn_3a_{18} + \Psi_{11}a_{23} + \Psi_{22}[a_{22} - k^2n_3^2a_{21} - c^2k^2a_{24}] = 0,$$

$$-U_2ikn_3a_{10} + U_3ik[-n_2a_{19} + k^2n_2n_3^2a_{20} + c^2k^2n_2a_{14}] + \Psi_{23}[a_{28} - k^2n_3^2a_{27} - c^2k^2a_{24}] = 0.$$

Let us first examine system II. On setting $n_2 = 0$, system II splits into two groups of equations. In the first group we have two equations for U_2, Ψ_{23} and get a transverse wave (macroscopically) propagating in the direction of the fibres. In the second group we have three equations for U_3, Ψ_{11}, Ψ_{22} and the wave is longitudinal, propagating in the direction of the fibres. On setting $n_3 = 0$, system II splits again. In the first group there are three equations for U_2, Ψ_{11}, Ψ_{22} which represent a longitudinal wave propagating normal to the fibres. The other group contains two equations for U_3, Ψ_{23} and we get a transverse wave polarized in the x_2x_3-plane, travelling normal to the fibres.

On setting $n_3 = 0$ in system I we obtain three equations for U_1, Ψ_{12}, Ψ_{21} and this wave is transverse polarized in the x_1x_2-plane, travelling normal to the fibres. The substitution $n_2 = 0$ yields two equations for U_1, Ψ_{13}. Because of symmetry about the x_3-axis these equations are the same as those for U_2, Ψ_{23} obtained from system II with $n_2 = 0$.

All the systems of equations for the amplitudes are homogeneous. We get non-trivial solutions only if the characteristic determinants of the systems are zero. These conditions give us the sought-for dispersive relations. For a transverse wave propagating in the direction of the fibres the dispersion relation between c and k is

$$c^4k^4a_{12}a_{24} + c^2[k^2(a_{12}a_{27} + a_3a_{24}) - a_{12}a_{28}] + k^2a_3a_{27} - (a_3a_{28} + a_{10}^2) = 0. \tag{4.1}$$

For a longitudinal wave in the direction of the fibres we get

$$c^6k^4a_{12}a_{24}^2 + c^4[k^4a_{24}(a_{15}a_{24} + 2a_{12}a_{21}) - k^2\cdot 2a_{12}a_{22}a_{24} + c^2[k^4\cdot a_{21}(a_{12}a_{21} + 2a_{15}a_{24}) - k^2\cdot 2\{a_{24}(a_{15}a_{22} + a_{18}^2) + a_{12}a_{21}a_{22}\} + a_{12}(a_{22}^2 - a_{23}^2)] + k^4a_{15}a_{21}^2 - k^2\cdot 2a_{21}(a_{15}a_{22} + a_{18}^2) + (a_{22} - a_{23})[a_{15}(a_{22} + a_{23}) + 2a_{18}^2] = 0. \tag{4.2}$$

We shall not state here the dispersion relations for the other waves but shall write instead the expressions of the phase velocities $^\circ c$ of the lowest mode at vanishing wave numbers for all the types of waves which we shall need later on.

(a) The transverse wave propagating in the direction of the fibres:
(4.1) yields

$$^\circ c^2 = \lim_{k\to 0} c^2 = -\frac{a_3a_{28} + a_{10}^2}{a_{12}a_{28}}. \tag{4.3}$$

(b) The longitudinal wave propagating in the direction of the fibres:
(4.2) yields

$$^\circ c^2 = -\frac{a_{15}(a_{22} + a_{23}) + 2a_{18}^2}{a_{12}(a_{22} + a_{23})}. \tag{4.4}$$

(c) The longitudinal wave propagating normal to the fibres:

$$^{\circ}c^2 = \frac{a_1(a_{22}^2 - a_{23}^2) - 2a_7a_8a_{23} + a_{22}(a_7^2 + a_8^2)}{a_{12}(a_{23}^2 - a_{22}^2)}. \tag{4.5}$$

(d) The transverse wave polarized in the x_1x_2-plane propagating normal to the fibres:

$$^{\circ}c^2 = \frac{a_2(a_{25}^2 - a_{26}^2) - 2a_8a_9a_{26} + a_{25}(a_8^2 + a_9^2)}{a_{12}(a_{26}^2 - a_{25}^2)}. \tag{4.6}$$

(e) The transverse wave polarized in the x_2x_3-plane propagating normal to the fibres:

$$^{\circ}c^2 = -\frac{a_{16}a_{28} + a_{19}^2}{a_{12}a_{28}}. \tag{4.7}$$

(f) The plane wave polarized in the x_2x_3-plane propagating in the direction parallel to the x_2x_3-plane. For such a wave we have

$$U_1 = \Psi_{12} = \Psi_{21} = \Psi_{13} = 0, \quad n_1 = 0.$$

For $^{\circ}c$ of this wave system II yields the following condition:

$$^{\circ}c^4\alpha_1 + {}^{\circ}c^2(n_2^2\alpha_2 + n_3^2\alpha_3) + n_2^4\alpha_4 + n_3^4\alpha_5 + n_2^2n_3^2\alpha_6 = 0, \tag{4.8}$$

where

$$\alpha_1 = a_{12}^2a_{28}(a_{22}^2 - a_{23}^2),$$

$$\alpha_2 = a_{12}(a_{22}^2 - a_{23}^2)[a_{19}^2 + a_{28}(a_1 + a_{16})] + a_{12}a_{28}[a_{22}(a_7^2 + a_8^2) - 2a_7a_8a_{23}],$$

$$\alpha_3 = a_{12}(a_{22}^2 - a_{23}^2)[a_{10}^2 + a_{28}(a_3 + a_{15})] + 2a_{12}a_{18}^2a_{28}(a_{22} - a_{23}),$$

$$\alpha_4 = (a_{19}^2 + a_{16}a_{28})[a_1(a_{22}^2 - a_{23}^2) + a_{22}(a_7^2 + a_8^2) - 2a_7a_8a_{23}],$$

$$\alpha_5 = (a_{22} - a_{23})(a_3a_{28} + a_{10}^2)[a_{15}(a_{22} + a_{23}) + 2a_{18}^2],$$

$$\alpha_6 = (a_{22}^2 - a_{23}^2)[a_{16}(a_{10}^2 + a_3a_{28}) + a_3(a_{19}^2 + a_{16}a_{28}) - a_5^2a_{28}] + 2(a_{23} - a_{22})a_{18}(a_7 + a_8)$$
$$\times (a_{10}a_{19} + a_5a_{28}) + a_{28}[(a_7^2 + a_8^2)(a_{18}^2 + a_{15}a_{22}) - 2a_7a_8(a_{18}^2 + a_{15}a_{23})].$$

Because $U_2 \neq 0$, $U_3 \neq 0$ such a wave is neither longitudinal nor transverse for $n_2 \neq 0$, $n_3 \neq 0$.

Figure 2 shows the lowest modes of the dispersion curves for the transverse wave propagating in the direction of the fibres. In place of c, k there are plotted there the dimensionless quantities β, ξ defined by

$$\beta = \frac{c}{\left(\frac{\mu_2}{\rho_2}\right)^{1/2}}, \quad \xi = kr_1.$$

The solid lines in Fig. 2 correspond to (4.1). a_{ij} in (4.1) are given by (3.14). The curves are drawn for the special choice of $\eta^2 = 0{\cdot}6$, $\vartheta = 3$, $\nu_1 = \nu_2 = \nu = 0{\cdot}3$, and for two values of γ: $\gamma = 10$, $\gamma = 100$. Here we write

$$\vartheta = \frac{\rho_1}{\rho_2}, \quad \nu_\alpha = \frac{\lambda_\alpha}{2(\lambda_\alpha + \mu_\alpha)}, \quad \alpha = 1, 2; \ \gamma = \frac{\mu_1}{\mu_2}.$$

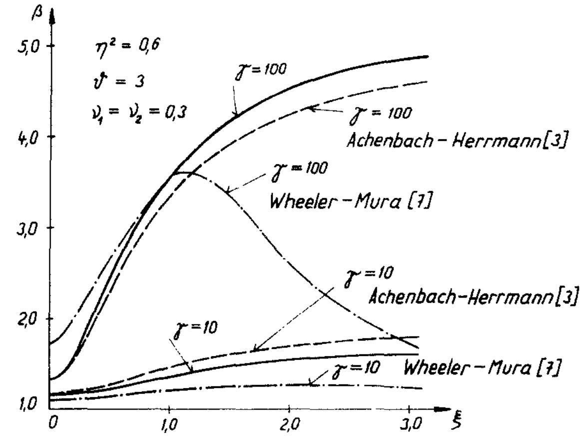

Fig. 2. Dispersion curves for transverse waves in the direction of fibres.

In the above, ν_1 and ν_2 are Poisson's ratios of the fibres and of the matrix, respectively. The dashed lines correspond to the theory of Achenbach and Herrmann[3] for randomly arranged fibres. The dot-and-dashed lines are taken from [7]. Using the Ritz method, Wheeler and Mura[7] give approximate curves for a fibre reinforced material with the fibres arranged in a square array. Figure 3 shows the lowest modes of the dispersion curves for the longitudinal wave propagating in the direction of the fibres. The solid lines correspond to (4.2) of this paper, the dot-and-dashed lines are taken from [7]. In the theory evolved in [3] this kind of wave is non-dispersive.

Figures 2 and 3 show a good agreement between all the curves for small wave numbers, i.e. for long wave lengths. A certain discrepancy in Fig. 2 for $k \to 0$ between the approximate curves taken from [7] on the one hand, and the curves of the present effective stiffness theory and those of the theory of [3] on the other hand, should be ascribed to the circumstance that a square array of fibres was considered in [7]. Unfortunately, the exact dispersion curves are not at our disposal.

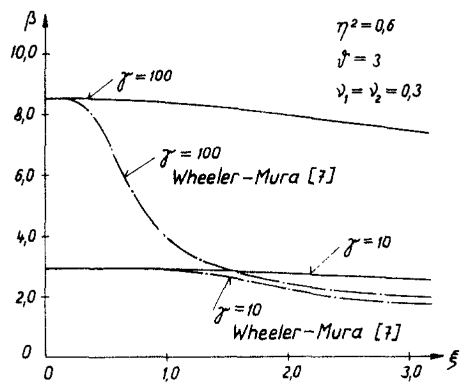

Fig. 3 Dispersion curves for longitudinal waves in the direction of fibres.

5. PROPAGATION OF PLANE HARMONIC WAVES IN A HOMOGENEOUS TRANSVERSELY ISOTROPIC MATERIAL

Propagation of plane harmonic waves in a homogeneous transversely isotropic material was examined by Postma[4]. We shall show that the structure of these waves in this continuum is the same as their structure in the effective stiffness theory presented in the foregoing paragraphs.

Let us write the constitutive equations of a transversely isotropic medium axially symmetric about the x_3-axis in the form

$$\tau_{11} = C_{11}\epsilon_{11} + C_{12}\epsilon_{22} + C_{13}\epsilon_{33},$$

$$\tau_{22} = C_{12}\epsilon_{11} + C_{11}\epsilon_{22} + C_{13}\epsilon_{33},$$

$$\tau_{33} = C_{13}\epsilon_{11} + C_{13}\epsilon_{22} + C_{33}\epsilon_{33},$$

$$\tau_{13} = 2C_{44}\epsilon_{13}, \quad \tau_{23} = 2C_{44}\epsilon_{23}, \quad \tau_{12} = C_{66}\epsilon_{12},$$

with

$$C_{66} = C_{11} - C_{12}.$$

There are five independent constants, C_{11}, C_{12}, C_{13}, C_{33} and C_{44}. For the wave

$$u_i = U_i e^{ik(n_2x_2+n_3x_3-ct)}$$

the displacement equations of motion lead to the following equations of the amplitudes U_i:

$$\left(\bar{\rho}c^2 - \frac{1}{2}C_{66}n_2^2 - C_{44}n_3^2\right)U_1 = 0, \tag{5.1}$$

$$(\bar{\rho}c^2 - C_{11}n_2^2 - C_{44}n_3^2)U_2 - (C_{13} + C_{44})n_2n_3U_3 = 0, \tag{5.2}$$

$$-(C_{13} + C_{44})n_2n_3U_2 + (\bar{\rho}c^2 - C_{44}n_2^2 - C_{33}n_3^2)U_3 = 0. \tag{5.3}$$

In the above $\bar{\rho}$ stands for the mass density. (5.1) corresponds to system I, (5.2) and (5.3) correspond to system II of Section 4. Using the same procedure as in Section 4 we get the following waves and the corresponding phase velocities c:

(a) The transverse wave propagating in the direction of the fibres:

$$c^2 = \frac{C_{44}}{\bar{\rho}}.$$

(b) The longitudinal wave propagating in the direction of the fibres:

$$c^2 = \frac{C_{33}}{\bar{\rho}}.$$

(c) The longitudinal wave propagating normal to the fibres:

$$c^2 = \frac{C_{11}}{\bar{\rho}}.$$

(d) The transverse wave polarized in the x_1x_2-plane propagating normal to the fibres:

$$c^2 = \frac{C_{66}}{2\bar{\rho}}.$$

(e) The transverse wave polarized in the x_2x_3-plane propagating normal to the fibres:

$$c^2 = \frac{C_{44}}{\bar{\rho}}.$$

(f) The transverse wave polarized in the x_2x_3-plane propagating in the direction parallel to the x_2x_3-plane:

$$\bar{\rho}^2c^4 - \bar{\rho}c^2[n_2^2(C_{11}+C_{44}) + n_3^2(C_{33}+C_{44})] + n_2^4C_{11}C_{44} + n_3^4C_{33}C_{44} + n_2^2n_3^2[C_{44}^2 + C_{11}C_{33} - (C_{13}+C_{44})^2] = 0.$$

The effective stiffness theory presented in this paper is a long-wave approximation of the dynamical behaviour of a fibre reinforced composite. For the statical problems we replace the composite by a homogeneous transversely isotropic medium. In order to obtain the material constants we compare the phase velocities of the non-dispersive waves (a–f) in the homogeneous transversely isotropic material with the phase velocities $°c$ at vanishing wave numbers in (4.3) to (4.8) for the corresponding waves (a–f) in the effective stiffness model. The result is

$$C_{44} = \frac{a_{16}a_{28} + a_{19}^2}{a_{28}} = \frac{a_3a_{28} + a_{10}^2}{a_{28}},$$

$$C_{33} = \frac{a_{15}(a_{22}+a_{23}) + 2a_{18}^2}{a_{22}+a_{23}},$$

$$C_{11} = \frac{a_1(a_{22}^2 - a_{23}^2) - 2a_7a_8a_{26} + a_{25}(a_8^2 + a_9^2)}{a_{22}^2 - a_{23}^2},$$

$$C_{66} = C_{11} - C_{12} = 2\frac{a_2(a_{25}^2 - a_{26}^2) - 2a_8a_9a_{26} + a_{25}(a_8^2 + a_9^2)}{a_{25}^2 - a_{26}^2},$$

$$C_{13} + C_{44} = \frac{(a_5a_{28} + a_{10}a_{19})(a_{22}+a_{23}) + a_{18}a_{28}(a_7+a_8)}{a_{28}(a_{22}+a_{23})}. \tag{5.4}$$

On substituting for a_{ij} from (3.14) we finally arrive at the sought-for moduli

$$C_{44} = \frac{\eta^2(\gamma-1)(4V+1) + 4V}{\eta^2(\gamma-1) + 4V}\mu_2,$$

$$C_{33} = \left\{\eta^2(\gamma\delta_1 - \delta_2) + \delta_2 - \frac{\eta^4(\gamma\epsilon_1 - \epsilon_2)^2}{\eta^2[\gamma(\delta_1-1) - (\delta_2-1)] + 2V\delta_2}\right\}\mu_2,$$

$$C_{13} = \left\{(\delta_2 - 2) + \frac{2\eta^2V\delta_2(\gamma\epsilon_1 - \epsilon_2)}{\eta^2[\gamma(\delta_1-1) - (\delta_2-1)] + 2V\delta_2}\right\}\mu_2,$$

$$C_{11} = \left\{\delta_2 + V(3\delta_2+1) - \frac{V^2\langle\eta^2[(\gamma-1)(3\delta_2+1)(\delta_2-1) + (\gamma\delta_1-\delta_2)(\delta_2+1)^2] + 2V\delta_2(3\delta_2+1)(\delta_2+1)\rangle}{[\eta^2(\gamma-1) + V(\delta_2+1)][\eta^2\langle\gamma(\delta_1-1) - (\delta_2-1)\rangle + 2V\delta_2]}\right\}\mu_2,$$

$$C_{66} = C_{11} - C_{12} = 2\left\{1 + V(\delta_2+1) - \frac{V^2(\delta_2+1)^2}{\eta^2(\gamma-1) + V(\delta_2+1)}\right\}\mu_2. \tag{5.5}$$

Here we write

$$\delta_\alpha = \frac{2(1-\nu_\alpha)}{1-2\nu_\alpha}, \quad \epsilon_\alpha = \frac{2\nu_\alpha}{1-2\nu_\alpha}, \quad \alpha = 1, 2.$$

6. CONCLUDING REMARKS

There remains the open question of what is generally the relation between the moduli (5.5) which we get for vanishing wave numbers, and the effective moduli which are a geometrically weighted average of the properties of the constituents. For a laminated medium we know the exact effective moduli and the exact dispersive curves, and in this case the two systems of moduli are equal.

For the case of laminated composites a higher approximation of the effective stiffness theory was elaborated in [8], with the stress vector continuous at the layer interfaces. This second-order approximation provided a substantially better approximation to the exact elasticity solution for shorter wave lengths but the phase velocities $^\circ c$ at vanishing wave numbers were the same as in the first approximation and thus the same as the exact $^\circ c$ [1].

The effective stiffness theory presented herein for fibre reinforced composites is the simplest approximation involving continuity of the displacements at the interfaces while leaving the stress vector discontinuous. We do not know the exact $^\circ c$ for this case. As the displacement (2.6)–(2.7) is kinematically admissible the moduli (5.5) should be larger or equal to those obtained from the exact $^\circ c$.

For a fibre reinforced composite with a hexagonal layout of fibres, Hashin and Rosen[6] reported the lower and the upper bound of the effective moduli. The moduli defined by (5.5) lie within these bounds. For randomly distributed fibres[6] states the approximate magnitudes of C_{13}, C_{33}, C_{44} and $\frac{1}{2}(C_{11}+C_{12})$, for $\frac{1}{2}(C_{11}-C_{12})$ giving again the lower and the upper bound. It was found that C_{13}, C_{33}, C_{44} and $\frac{1}{2}(C_{11}+C_{12})$ calculated from (5.5) for $\gamma = 10$, $\gamma = 100$ and $\eta^2\epsilon\langle 0, 1\rangle$ are very close to the approximate effective moduli given in Ref. [6] for randomly distributed fibres. The difference is less than 1 per cent.

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